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CERTAIN DOUBLE COSET SPACES OF ALGEBRAIC GROUPS AND RATIONAL BOUNDARY COMPONENTS OF SYMMETRIC BOUNDED DOMAINS

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I

In part I we shall consider the problem of determining the order of double cosets $\Gamma\backslash G/P$, where G is a certain k-algebraic group, P is its k-parabolic subgroup and Γ is its arithmetic subgroup. A detailed discussion on the subject is found in [5].

Let k be an algebraic number field of finite degree, and K be either a quadratic extension of k or k itself, and σ the involution of K stabilizing each element of k. Let V be a finite dimensional vector space over K supplied with a non-degenerate k-bilinear form $F: V \times V \longrightarrow K$ such that $F(ax,by) = a^{\sigma}F(x,y)b$ for $a,b \in K$, $x,y \in V$ and that $F(x,y)^{\sigma} = eF(y,x)$, $e = \pm 1$.

We set $G = \{ g \in GL(V);\ F(g(x),g(y)) = F(x,y),\ x,y \in V \}$ and $G^1 = G \cap SL(V)$. Then the groups G and G^1 are k-algebraic groups.

Suppose that there exists a proper non-zero subspace W of V such that $F(w,w') = 0$ for all $w,w' \in W$ (i.e. W is a totally isotropic subspace of V). We set $G_W = \{ g \in G;\ g(W) = W \}$. This is a maximal k-parabolic subgroup of G.

Let $\mathcal{O}_K$ be the ring of integers in K and let L be an $\mathcal{O}_K$-lattice in V. We set $G_L = \{ g \in G;\ g(L) = L \}$. This is an arithmetic subgroup of G.

Similarly, we get a maximal k-parabolic subgroup G^1_W and an arithmetic subgroup G^1_L of G^1.

Now, given any subgroup H of G and $\mathcal{O}_K$-submodules X,Y of V, we write $X \underset{H}{\sim} Y$ if and only if there exists an element h of H such that $h(X) = Y$.

We denote the set of $\mathcal{O}_K$-submodules Y such that $X \underset{H}{\sim} Y$ by $(X)_H$. Then, the double coset space $G_L\backslash G/G_W$ is in a bijective correspondence with either one of the sets $(W)_G/\underset{G_L}{\sim}$, or $(L)_G/\underset{G_W}{\sim}$. Thus the problem of determining the order $|G_L\backslash G/G_W|$ is reduced to a certain classification problem of lattices. The determination of the order $|G_L^1\backslash G^1/G_W^1|$ is, to a great extent, reduced to the determination of $|G_L\backslash G/G_W|$.

Associated to the lattice L we have a fractional ideal $(\mu_0(L))$ in K generated by $F(x,y)$ for $x,y \in L$. The lattice L is called a ($\mu_0(L)$-)modular if $L = \{x \in V;\ F(x,L) \subset \mu_0(L)\}$.

Then we have the following decomposition theorem:

<u>Let L be an $\mathcal{J}$-modular lattice in V.</u> <u>Then there exist $\mathcal{O}_K$-ideals</u> $\mathfrak{a}_1,\dots,\mathfrak{a}_s$, <u>a basis</u> $\{w_1,\dots,w_s\}$ <u>of W, and elements</u> $w_1',\dots,w_s'$ <u>of V such that</u>

$$L = \sum_{i=1}^{s}(\mathfrak{a}_i^{-\sigma}\mathcal{J}w_i + \mathfrak{a}_i w_i') + L', \text{ where } \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots \supset \mathfrak{a}_s,$$

$w_i \in L$, $F(w_i,w_j') = \delta_{ij}$, $F(w_i',w_j') = m_i\delta_{ij}$ <u>for all</u> i,j.

In the above, when $m_i = 0$ for all i (e.g. when $e = -1$), it is easy to determine the order $G_L\backslash G/G_W$. When $e = 1$, it becomes necessary to investigate the properties of the submodule $S(\mathcal{O}_K) = \{N(x) + Tr(y);\ x,y \in \mathcal{O}_K\}$ of $\mathcal{O}_k$, and submodule $S(L,W,\mathfrak{a}) = \{F(ax,ax) + Tr(b);\ a \in \mathfrak{a}^{1}, x \in L', b \in \mathfrak{a}^{-1-\sigma}\mathcal{J}\}$ of the module $S(L,\mathfrak{a}) = \{F(ax,ax) + Tr(b);\ a \in \mathfrak{a}^{-1}, x \in L, b \in \mathfrak{a}^{-1-\sigma}\mathcal{J}\}$ for $\mathcal{O}_K$-ideals $\mathfrak{a}$. It can be shown that if K is a quadratic extension of k, then $S(\mathcal{O}_K) = \mathcal{O}_k$, and that the order $|S(L,\mathfrak{a})/S(L,W,\mathfrak{a})|$ is generally independent of the choice of the ideal $\mathfrak{a}$; we denote the order by $s(L,W)$.

The order $|G_L\backslash G/G_W|$ for an $\mathcal{J}$-modular lattice L can be evaluated in terms of $h(K)$ (= the class number of K), $h(L')$ (= G-class number of L'), $s(L,W)$ etc. Specifically, we have the following estimation:

1) When $K = k$ and $e = -1$, then $|G_L\backslash G/G_W| = h(k)$.

2) If $S(\mathcal{O}_K) = \mathcal{O}_k$, and $s(M,W) = 1$ for all M belonging to the same G-genus as L, then $|G_L\backslash G/G_W| \leq h(K)h(L')$, and if, moreover, all $\mathcal{J}$-modular lattices in V are G-equivalent, then $|G_L\backslash G/G_W| = h(K)h(L')$.

The latter case occurs, for example, in the following situations:

1) $K = k$, $\dim V$ is odd, $S(\mathcal{O}_k) = \mathcal{O}_k$, $h(k) = 1$,

2) K is a quadratic extension of k, $\dim_K V$ is odd, and every ideal class in K is represented by a σ-invariant ideal.

EXAMPLES:

1) $k = Q$, $K = Q(\sqrt{-1})$, $\dim_K V$ is odd and V has a basis $\{v_1,\dots,v_n\}$ such that $(F(v_i,v_j)) = \mathrm{diag.}(1_p,-1_q)$, and $L = \sum \mathcal{O}_K v_i$. In this case,

$$|G_L\backslash G/G_W| = h(L') \leq |G^1_L\backslash G^1/G^1_W| \leq 2h(L'),$$

$$h(L')\begin{cases} = 1 \text{ when } W^\perp/W \text{ is indefinite ([9]), or the rank of } L' < 5\ [4] \\ > 1 \text{ when the rank of } L' \geq 5, \\ = 2 \text{ when the rank of } L' = 5, \\ = 4 \text{ when the rank of } L' = 7. \end{cases}$$

2) $k = Q$, $K = Q(\sqrt{-p})$, $p \equiv 3 \bmod 4$, $\dim_K V$ is odd and V has a basis $\{v_1,\dots, v_n\}$ such that $(F(v_i,v_j)) = \mathrm{diag.}(1_{n-1},-1)$, and $L = \sum \mathcal{O}_K v_i$. Then

$$|G_L\backslash G/G_W| = |G^1_L\backslash G^1/G^1_W| = h(K).$$

II

We assume that G^1 is simply connected (hence, G^1 is either $SU(V,H)$ or $Sp(V,A)$). We assume further that the Lie group $(\mathcal{R}_{k/Q}(G^1))_R$ admits a maximal compact subgroup $\mathcal{K}$ such that $D = (\mathcal{R}_{k/Q}(G^1))_R/\mathcal{K}$ has the structure of a symmetric bounded domain (hence, k is totally real, and K is either k itself or a totally imaginary quadratic extension of k).

In this case, the subspace W corresponds to a rational boundary component $B(W)$ of $\bar{D}$, and conversely, for any rational boundary component of $\bar{D}$ there exists a totally isotropic subspace W' of V such that the boundary component may be written as $B(W')$ (cf. [1]); the dimension of such a subspace W' is determined by the given boundary component which we shall call the type of the boundary component. Let $\widetilde{B}(W)$ be the set of rational boundary components of $\bar{D}$ having the same type as $B(W)$. $\widetilde{B}(W)$ is a G^1-orbit space. The double coset space $G^1_L\backslash G^1/G^1_W$ is in a bijective correspondence with the set of G^1_L-orbits among $\widetilde{B}(W)$.

III

We ~~give~~ may make a remark concerning our previous work in [2] and [3]. Let $D^* = D \cup \{$rational boundary components of $D\}$ supplied with Satake topology, and let $V^* = G^1_L\backslash D^*$. Then V^* has the structure of a projective variety.

Consider a functor sending the category of Hermitian vector spaces (V,H) to the category of alternating vector spaces (V',A), where $V' = \mathcal{R}_{K/k}V$ and A is the "imaginary part" of H. This functor naturally induces a rational homomorphism sending $G^1 = SU(V,H)$ into $G' = Sp(V',A)$; lattices L in V naturally correspond to lattices L' in V'.

When L is modular and $\mu_0(L)$ is an ideal in k, then the corresponding lattice L' is maximal in V'. When, in general, L is $\mathcal{J}$-modular, the elementary divisors of L' may be explicitly described in terms of $\mathcal{J}$ if (2) is a prime ideal in k (cf. [6]).

Let D, D' be the symmetric bounded domains corresponding to G^1, G'. Assume that $(\mathcal{R}_{k/Q}\rho)(\mathcal{K}) \subset \mathcal{K}'$, then ρ induces a holomorphic imbedding of D into D' (cf. [7]); this ρ further induces a morphism of the variety

V^* into V'^* (cf [8]). (We have $\rho(G^1_L) \subset G^!_{L'}$.)

We may ask here, when are automorphic forms on D with respect to G_L ~~may be~~ extendable to automorphic forms on D' with respect to $G^!_{L'}$? The above I, II may be helpful ~~to~~ in considering this problem.

In particular, the field of rational functions $C(V^*)$, which is identified with the field of automorphic functions on D with respect to G^1_L, may be identified with a subfield of $C(\rho(V^*))$, and their relations may be described in terms of certain Galois cohomology group (cf. [2], [3]).

Especially, when $k = Q$, $K = Q(\sqrt{-p})$, $p \equiv 3 \mod 4$, $p > 3$, $\dim_K V$ is odd then $C(V^*) = C(\rho(V^*))$.

REFERENCES

1) W.L.Baily,Jr. and A.Borel, Compactification of arithmetic quotients of bounded symmetric domains, Ann. of Math. (3), 84 (1966), 442-528.

2) K.Iyanaga, Arithmetic of special unitary groups and their symplactic representations, J. Fac. Sci. Univ. Tokyo, 15 (1968), 35-69.

3) _______, Symplectic 表現と保型関数, 京大数理解析研究所講究録 41, 46-55

4) _______, Class numbers of definite Hermitian forms, J. Math. Soc. Japan, 21 (1969), 359-374.

5) _______, On certain double coset spaces of algebraic groups, J.Math. Soc. Japan, 23 (1971), 103-122.

6) _______, Arithmetic of Hermitian forms and related forms, to appear.

7) I.Satake, Holomorphic imbeddings of symmetric domains into a Siegel space, Amer.J.Math., 87 (1964), 425-461.

8) _______, A note on holomorphic imbeddings and compactifications of symmetric domains, Amer.J.Math. 90 (1968),231-247.

9) G.Shimura, Arithmetic of unitary groups, Ann. of Math., (2) 79 (1964), 369-409.